Probability Density Function Solution of Nonlinear Oscillators Subjected to Multiplicative Poisson Pulse Excitation on Velocity

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The stationary probability density function (PDF) solution of the stochastic responses is derived for nonlinear oscillators subjected to both additive and multiplicative Poisson white noises. The PDF solution is governed by the generalized Fokker–Planck–Kolmogorov (FPK) equation and obtained with the exponential-polynomial closure (EPC) method, which was originally proposed for solving the FPK equation. The extended EPC solution procedure is presented for the case of Poisson pulses in this paper. In order to evaluate the effectiveness of the solution procedure, nonlinear oscillators are investigated under multiplicative Poisson white noise excitation on velocity and additive Poisson white noise excitation. Both weakly and strongly nonlinear oscillators are considered, respectively. In the numerical analysis, both the unimodal and bimodal stationary PDFs of oscillator responses are obtained with the EPC method and Monte Carlo simulation. Compared with the simulation results, good agreement is achieved with the presented solution procedure in the case of the polynomial degree being 6, especially in the tail regions of the PDFs of the system responses. [DOI: 10.1115/1.4000385]

Keywords: nonlinear, oscillator, generalized FPK equation, probability density function, Poisson white noise

1 Introduction

Poisson white noise represents a discrete excitation process as a sequence of independent, identically distributed pulses arriving at random times. It has been applied to model the loadings induced by earthquakes, wind loads, shock waves, or traffic loads. Furthermore, some systems can be described with nonlinear oscillators excited by Poisson white noises. In the case of multiplicative excitation being Gaussian white noise, a well known Wong and Zakai [1] or Stratonovich [2] correction term is added to the drift term [1,2] when a physical differential equation is transferred to an Itô type stochastic differential equation. This modification term takes into account the local irregularity of the Brownian motion process and is widely adopted in the studies on Gaussian multiplicative excitation. Corrective terms are also introduced for multiplicative Poisson white noise excitations [3–12]. If the excitation is Poisson white noise, the probability density function (PDF) of system responses is governed by the generalized Fokker–Planck–Kolmogorov (FPK) equation [13,14]. In order to solve the generalized FPK equation, a perturbation scheme was adopted for the system with external excitation [13] and extended to the system with either external excitation or multiplicative excitation on displacement [14]. With the perturbation scheme, a suitable initial solution and a small perturbation parameter needs to be preliminarily determined. Subsequently, other approximation or numerical methods were also proposed for analyzing the oscillators under external Poisson white noise excitation, such as Petrov–Galerkin method [15] and cell-to-cell mapping (path integration) technique [16,17]. The effectiveness of these two approaches was briefly discussed in Ref. [18]. It is observed that the Petrov–Galerkin technique behaves well in the case that the mean arrival rate of Poisson white noise is high whereas the cell-to-cell mapping technique is suitable in the case that the mean arrival rate of Poisson white noise is low as stated in Ref. [18]. Finite difference method [19,20] was also employed to solve the generalized FPK equation in the case that the system excitation is external. With this method, the characteristic function is derived first and then the response PDF is obtained by an inverse numerical Fourier transformation of the characteristic function. The selection of an insufficient domain may result in inaccurate values of the PDF solution or response moments. Possible negative PDF value in the tail regions may also be obtained with the finite difference method. Recently, an inverse solution procedure was adopted to obtain the exact solution of some derived systems in some special cases [21–23]. The system functions are derived based on the prescribed PDF, which is highly restricted and difficult to apply directly in practice. Besides the above methods for obtaining the PDF solution to the generalized FPK equation, some approaches were also developed for obtaining the statistical moments of the responses of the systems excited by Poisson white noise. Equivalent linearization (EQL) procedures for the systems with external Poisson excitation [24–27] were extensively investigated and utilized for obtaining the statistical moments. When impulse arrival rates are high and the system nonlinearity is weak, the EQL procedures can be appropriate for the second moment estimation. The response moments of the systems under multiplicative Poisson excitation were investigated with various linearization techniques [28]. The linearization techniques originally developed for the systems under multiplicative Gaussian excitation were extended and examined for the systems under multiplicative Poisson excitation. It indicates that the results obtained with different linearization techniques are at the same level of accuracy in the case of purely external excitation whereas the obtained results are incon-

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sistent in the case of multiplicative excitation. The cumulant-neglect closure procedure was also utilized to analyze the systems under external Poisson white noise [29,30] and the systems under multiplicative Poisson white noise [5]. The response moments obtained with the cumulant-neglect closure procedure can be good in the case that the system nonlinearity is slight and the impulse arrival rate is high.

From the above, it is seen that finding the PDF solution of the system responses is a challenge problem if the excitations in the oscillator are Poisson white noises. The obtainable exact stationary PDF solutions are highly restricted, or most various numerical methods are limited to the systems under external Poisson excitation. As a result, the effectiveness of the approximation methods relies heavily on the level of system nonlinearity or the value of impulse arrival rate of Poisson white noise. Another challenge is that the PDF solutions are less investigated for the systems under multiplicative Poisson excitation, especially for the systems under multiplicative excitation on velocity. In this paper, the exponential-polynomial closure (EPC) method originally developed for the systems under Gaussian excitations [31–36] is extended for the approximate PDF solution of nonlinear oscillators under both additive and multiplicative Poisson white noise excitations. In order to evaluate the effectiveness of this method for the oscillators under multiplicative Poisson excitation on velocity, an extended solution procedure is formulated and presented for the approximate solution of the generalized FPK equation. Both weakly and highly nonlinear oscillators under both additive and multiplicative Poisson excitations are analyzed. In the numerical analysis, the oscillators with either unimodal or bimodal PDF solutions are analyzed with the EPC method and Monte Carlo simulation. Compared with the simulation results, good agreement is achieved with the presented solution procedure, especially in the tail regions of the PDFs of the oscillator responses.

2 Problem Formulation

Consider the following nonlinear stochastic oscillator:

$$\ddot{X} + h_0(X,\dot{X}) = h_j(X,\dot{X})W_j(t) \tag{1}$$

where $X \in \Re$ and $\dot{X} \in \Re$ are stochastic processes, $\Re$ denotes real space; h_0 and h_j are functions of X and $\dot{X}$. These functions can be either linear or nonlinear and their functional forms are assumed to be deterministic; $W_j(t)$ is the jth process of random excitation and all $W_j(t)$ are independent of each other. Furthermore, Poisson white noise excitation is employed and formulated as

$$W_j(t) = \sum_{k=1}^{N(T)} Y_{jk}\delta(t-\tau_k) \tag{2}$$

where $N(T)$ is the total number of pulses that arrive in the time interval $(-\infty, T]$. Y_{jk} is an impulse amplitude of the kth pulse arriving at time τ_k for $W_j(t)$. $\delta(t)$ is the Dirac delta function. In this paper, $N(T)$ is assumed to yield the Poisson law with a constant average impulse arrival rate λ_j. The impulse amplitudes Y_{jk} are independent identically distributed random variables with zero mean and also independent of the pulse arrival time τ_k. When the impulse arrival rate is constant and the impulse amplitudes are identically distributed, the process becomes stationary [10]. Setting $X=x_1$ and $\dot{X}=x_2$, Eq. (1) can be expressed as

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h_0(x_1,x_2) + h_j(x_1,x_2)W_j(t) \end{cases} \tag{3}$$

The response vector $\{x_1,x_2\}^T$ is Markovian and the PDF of the responses is governed by the following generalized FPK equation [14]:

$$\begin{aligned} \frac{\partial p}{\partial t} = &-x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left(h_0 - \frac{1}{2}\lambda_j E[Y_j^2]h_j\frac{\partial h_j}{\partial x_2}\right)p\right\} \\ &+ \frac{1}{2!}\lambda_j E[Y_j^2]\frac{\partial^2}{\partial x_2^2}(h_j^2 p) - \frac{1}{3!}\lambda_j E[Y_j^3]\frac{\partial^3}{\partial x_2^3}(h_j^3 p) \\ &+ \frac{1}{4!}\lambda_j E[Y_j^4]\frac{\partial^4}{\partial x_2^4}(h_j^4 p) + \ldots \end{aligned} \tag{4}$$

where $E[\bullet]$ denotes the expectation of $(\bullet)$. Furthermore, if only the stationary PDF solution is considered, the term on the left side of Eq. (4) vanishes and the generalized FPK equation is reduced to be

$$\begin{aligned} &-x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left(h_0 - \frac{1}{2}\lambda_j E[Y_j^2]h_j\frac{\partial h_j}{\partial x_2}\right)p\right\} + \frac{1}{2!}\lambda_j E[Y_j^2]\frac{\partial^2}{\partial x_2^2}(h_j^2 p) \\ &- \frac{1}{3!}\lambda_j E[Y_j^3]\frac{\partial^3}{\partial x_2^3}(h_j^3 p) + \frac{1}{4!}\lambda_j E[Y_j^4]\frac{\partial^4}{\partial x_2^4}(h_j^4 p) + \ldots = 0 \end{aligned} \tag{5}$$

Because there are an infinite number of higher order derivative terms in Eq. (5), it is difficult to obtain its exact solution if it is possible. Here only the terms up to the fourth order derivative are retained for analysis in view that the contribution of high order terms is small to the whole equation. Hence

$$\begin{aligned} &-x_2\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left(h_0 - \frac{1}{2}\lambda_j E[Y_j^2]h_j\frac{\partial h_j}{\partial x_2}\right)p\right\} + \frac{1}{2!}\lambda_j E[Y_j^2]\frac{\partial^2}{\partial x_2^2}(h_j^2 p) \\ &- \frac{1}{3!}\lambda_j E[Y_j^3]\frac{\partial^3}{\partial x_2^3}(h_j^3 p) + \frac{1}{4!}\lambda_j E[Y_j^4]\frac{\partial^4}{\partial x_2^4}(h_j^4 p) = 0 \end{aligned} \tag{6}$$

It is assumed that the stationary PDF $p(x_1,x_2)$ of the responses of the random oscillator is subjected to the following conditions:

$$\begin{cases} p(x_1,x_2) \geq 0 \quad x_1,x_2 \in \Re^2 \\ \lim\limits_{x_i\to\pm\infty} p(x_1,x_2) = 0 \quad i=1,2 \\ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} p(x_1,x_2)dx_1dx_2 = 1 \end{cases} \tag{7}$$

These requirements should be fulfilled by the PDF solution of the oscillator expressed by Eq. (1). Generally, the exact solution of Eq. (6) is not obtainable. Therefore, an approximate PDF subjected to condition (7) needs to be formulated. Here the approximate PDF is expressed as an exponential-polynomial function of state variables with the EPC method [31–36]. The approximate PDF solution $\tilde{p}(x_1,x_2;\mathbf{a})$ to Eq. (6) is assumed to be

$$\tilde{p}(x_1,x_2;\mathbf{a}) = ce^{Q_n(x_1,x_2;\mathbf{a})} \tag{8}$$

where c is a normalization constant. $\mathbf{a}$ is an unknown parameter vector containing N_p entries. The polynomial $Q_n(x_1,x_2;\mathbf{a})$ is expressed as

$$Q_n(x_1,x_2;\mathbf{a}) = \sum_{i=1}^{n}\sum_{j=0}^{i} a_{ij}x_1^{i-j}x_2^j \tag{9}$$

which is an n degree polynomial in x_1 and x_2. To fulfill condition (7), it is required that

$$Q_n(x_1,x_2;\mathbf{a}) = -\infty \quad x_1,x_2 \notin \Omega \tag{10}$$

where $\Omega=[m_1-c_1\sigma_1, m_1+d_1\sigma_1]\times[m_2-c_2\sigma_2, m_2+d_2\sigma_2] \in \Re^2$ in which m_i and σ_i denote the mean values and standard deviations of x_i, respectively, $(i=1,2)$. c_i and d_i are positive constants. The values of c_i and d_i can be selected such that $m_i-c_i\sigma_i$ and $m_i+d_i\sigma_i$ locate in the tails of the PDF of x_i. This means that the approximate PDF is assumed to vanish beyond Ω.

Generally, the generalized FPK Eq. (6) cannot be fulfilled exactly with $\tilde{p}(x_1,x_2;\mathbf{a})$, because $\tilde{p}(x_1,x_2;\mathbf{a})$ is only an approximation of $p(x_1,x_2)$, and the number of unknown parameters N_p is always limited in practice. Substituting $\tilde{p}(x_1,x_2;\mathbf{a})$ for $p(x_1,x_2)$ leads to the following residual error:

$$\Delta(x_1,x_2;\mathbf{a}) = -x_2\frac{\partial\tilde{p}}{\partial x_1} + \frac{\partial}{\partial x_2}\left\{\left(h_0 - \frac{1}{2}\lambda_j E[Y_j^2]h_j\frac{\partial h_j}{\partial x_2}\right)\tilde{p}\right\} + \frac{1}{2!}\lambda_j E[Y_j^2]\frac{\partial^2}{\partial x_2^2}(h_j^2\tilde{p}) - \frac{1}{3!}\lambda_j E[Y_j^3]\frac{\partial^3}{\partial x_2^3}(h_j^3\tilde{p}) + \frac{1}{4!}\lambda_j E[Y_j^4]\frac{\partial^4}{\partial x_2^4}(h_j^4\tilde{p}) \quad (11)$$

By substituting Eq. (8) and the derivatives of $\tilde{p}(x_1,x_2;\mathbf{a})$ with respect to x_1 and x_2 given by Eq. (A1) in the Appendix into Eq. (11), the residual error can be expressed as follows:

$$\Delta(x_1,x_2;\mathbf{a}) = F(x_1,x_2;\mathbf{a})\tilde{p}(x_1,x_2;\mathbf{a}) \quad (12)$$

where $F(x_1,x_2;\mathbf{a})$ is a function of $Q_n(x_1,x_2;\mathbf{a})$. Because $\tilde{p}(x_1,x_2;\mathbf{a}) \neq 0$, the only possibility for $\tilde{p}(x_1,x_2;\mathbf{a})$ to satisfy Eq. (6) is $F(x_1,x_2;\mathbf{a})=0$. However, $F(x_1,x_2;\mathbf{a}) \neq 0$ in general because $\tilde{p}(x_1,x_2;\mathbf{a})$ is only an approximation of $p(x_1,x_2)$. In this case, another set of mutually independent functions $H_s(x_1,x_2)$ that span space $\mathfrak{R}^{N_p}$ can be introduced to make the projection of $F(x_1,x_2;\mathbf{a})$ on $\mathfrak{R}^{N_p}$ vanish, which leads to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(x_1,x_2;\mathbf{a})H_s(x_1,x_2)dx_1dx_2 = 0 \quad (13)$$

This means that the reduced generalized FPK equation is fulfilled with $\tilde{p}(x_1,x_2;\mathbf{a})$ in the weak sense of integration if $F(x_1,x_2;\mathbf{a})H_s(x_1,x_2)$ is integrable in $\mathfrak{R}^{N_p}$. Selecting $H_s(x_1,x_2)$ as

$$H_s(x_1,x_2) = x_1^{k-l}x_2^l f_1(x_1)f_2(x_2) \quad (14)$$

where $k=1,2,\ldots,n$; $l=0,1,2,\ldots,k$ and $s=\frac{1}{2}(k+2)(k-1)+l+1$; N_p nonlinear algebraic equations in terms of N_p unknown parameters can be formulated. The algebraic equations can be solved with any available method to determine the parameters. Numerical experience shows that a convenient and effective choice for $f_1(x_1)$ and $f_2(x_2)$ is the modified PDF obtained with the EQL method under Gaussian excitation with the intensity $\lambda E[Y^2]$ as follows:

$$f_1(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{x_1^2}{2\sigma_1^2}\right\} \quad (15)$$

$$f_2(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{x_2^2}{2\sigma_2^2}\right\} \quad (16)$$

where $\sigma_1=\sigma_{x_1}$ and $\sigma_2=r\sigma_{x_2}$. r is a positive modification factor, which controls the convergence of the solution procedure. Because of the particular choice for $f_1(x_1)$ and $f_2(x_2)$, the integration in Eq. (13) can be easily calculated by taking into account the relationships between higher and lower order moments of Gaussian random variables.

3 Illustrative Example

Consider the following oscillator excited by both additive and multiplicative Poisson white noise excitations:

$$\ddot{X} + 2\zeta\omega_0\dot{X} + \omega_0^2(\alpha X + \varepsilon_1 X^3) = W_1(t) + \varepsilon_2\dot{X}W_2(t) \quad (17)$$

For this oscillator, $h_0=2\zeta\omega_0\dot{X}+\omega_0^2(\alpha X+\varepsilon_1 X^3)$, $h_1=1.0$, and $h_2=\varepsilon_2\dot{X}$. $W_1(t)$ and $W_2(t)$ yield the form of Eq. (2) and are independent of each other. With these expressions, the residual error expressed by Eq. (11) is given by Eq. (A2) in the Appendix.

In this paper, the pattern of the PDF distribution and the influence of the oscillator nonlinearity on the effectiveness of the presented solution procedure are investigated with both additive and multiplicative Poisson white noise excitations being considered. The values of α and ε_1 are listed in Table 1. The values of other parameters are kept unchanged for all the oscillators in the following examples and they are given as: $\zeta=0.05$, $\omega_0=1$, $\varepsilon_2=0.1$, $\lambda_1 E[Y^2]=1$, $\lambda_2 E[Y'^2]=0.1$, and $\lambda_1=\lambda_2=1$. Y and Y' are Gaussian with zero mean. Meanwhile, Monte Carlo simulation (MCS) with a sample size 2×10^7 is also conducted for the simulation results.

Table 1 Parameters in the illustrative example

Case	α	ε_1	Remarks
1	1	0.1	Unimodal PDF+weak nonlinearity
2	1	1	Unimodal PDF+strong nonlinearity
3	−1	0.1	Bimodal PDF+weak nonlinearity
4	−1	1	Bimodal PDF+strong nonlinearity

3.1 Unimodal PDF and Weakly Nonlinear Restoring Force. In this case, $\alpha=1$ and $\varepsilon_1=0.1$. The nonlinearity of the restoring force in the oscillator is weak and there is only one peak in the PDF of displacement.

Figures 1(*a*)–1(*d*) present the comparison of the PDFs of displacement and velocity obtained with MCS and the EPC method ($n=2$, $n=6$). The numerical results show that the PDF obtained with EPC ($n=2$) is close to that obtained with the EQL method or Gaussian closure method in the case of Gaussian excitations with the intensities $\lambda_1 E[Y^2]$ and $\lambda_2 E[Y'^2]$, respectively. Figure 1(*a*) shows that the PDF of displacement obtained with the EPC method ($n=2$) or the EQL method deviates significantly from the simulation result. This deviation is more pronounced in the tail region as shown in Fig. 1(*b*). When the polynomial degree increases to 6, the obtained PDF of displacement with the EPC method ($n=6$) is in good agreement with that obtained with Monte Carlo simulation.

The behavior of the PDFs of velocity is shown in Figs. 1(*c*) and 1(*d*). It can be seen from Fig. 1(*c*) that the results from both EPC ($n=2$) and EPC ($n=6$) are close to the simulation result. Improvement can still be observed from the PDF obtained with EPC ($n=6$) as shown in Fig. 1(*d*). This means that the PDF of velocity looks close to Gaussian PDF but the non-Gaussian behavior and improvement of the PDF obtained with EPC ($n=6$) can be observed in the tail regions of the PDFs.

3.2 Unimodal PDF and Strongly Nonlinear Restoring Force. Consider the oscillator with strong nonlinearity and unimodal PDFs of responses in this case and $\alpha=1$ and $\varepsilon_1=1$. Figures 2(*a*)–2(*d*) present the comparison of the PDFs of displacement and velocity obtained with Monte Carlo simulation and the EPC method ($n=2$, $n=6$). Compared with the case of weakly nonlinear oscillator shown in Figs. 1(*a*)–1(*d*), the PDF obtained with EPC ($n=2$), in the case of strong nonlinearity, differs further from the simulation result, especially in the tail regions as shown in Fig. 2(*b*). On the other hand, the PDFs and logarithmic PDFs obtained with the EPC method ($n=6$) agree well with the simulation results for both displacement and velocity.

3.3 Bimodal PDF and Weakly Nonlinear Restoring Force. If $\alpha<0$ and $\varepsilon_1>0$, the PDF of displacement of the oscillator can have a bimodal pattern. The oscillator with the bimodal PDF of displacement is considered in this case by setting $\alpha=-1$ and $\varepsilon_1=0.1$. Figures 3(*a*)–3(*d*) show the results obtained with Monte Carlo simulation and the EPC method ($n=2$, $n=6$). As Fig. 3(*a*) shows, the PDF of displacement exhibits a distinct bimodal distribution. The PDF of displacement obtained with EPC ($n=2$) or the EQL method deviates a lot from the simulation result as shown in Fig. 3(*a*) and Fig. 3(*b*). On the other hand, the PDF of displacement obtained with EPC ($n=6$) also exhibits a bimodal

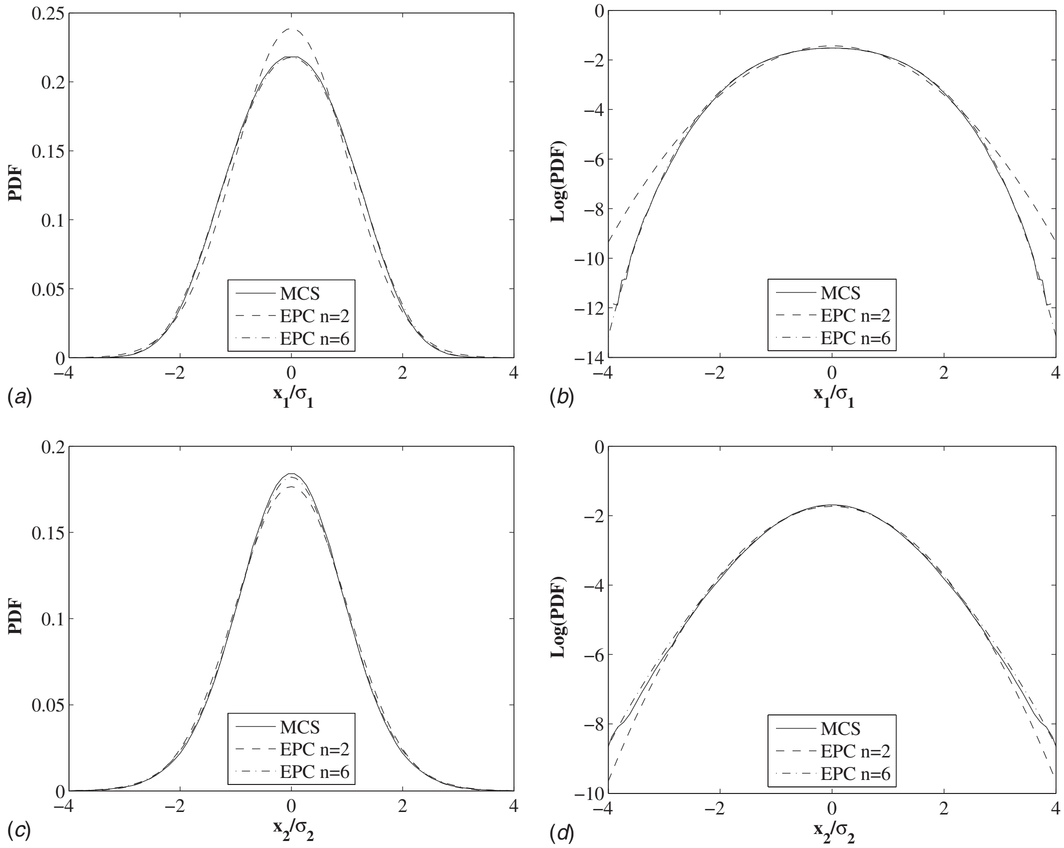

Fig. 1 Comparison of PDFs in Sec. 3.1: (*a*) PDFs of displacement, (*b*) logarithmic PDFs of displacement, (*c*) PDFs of velocity, and (*d*) logarithmic PDFs of velocity

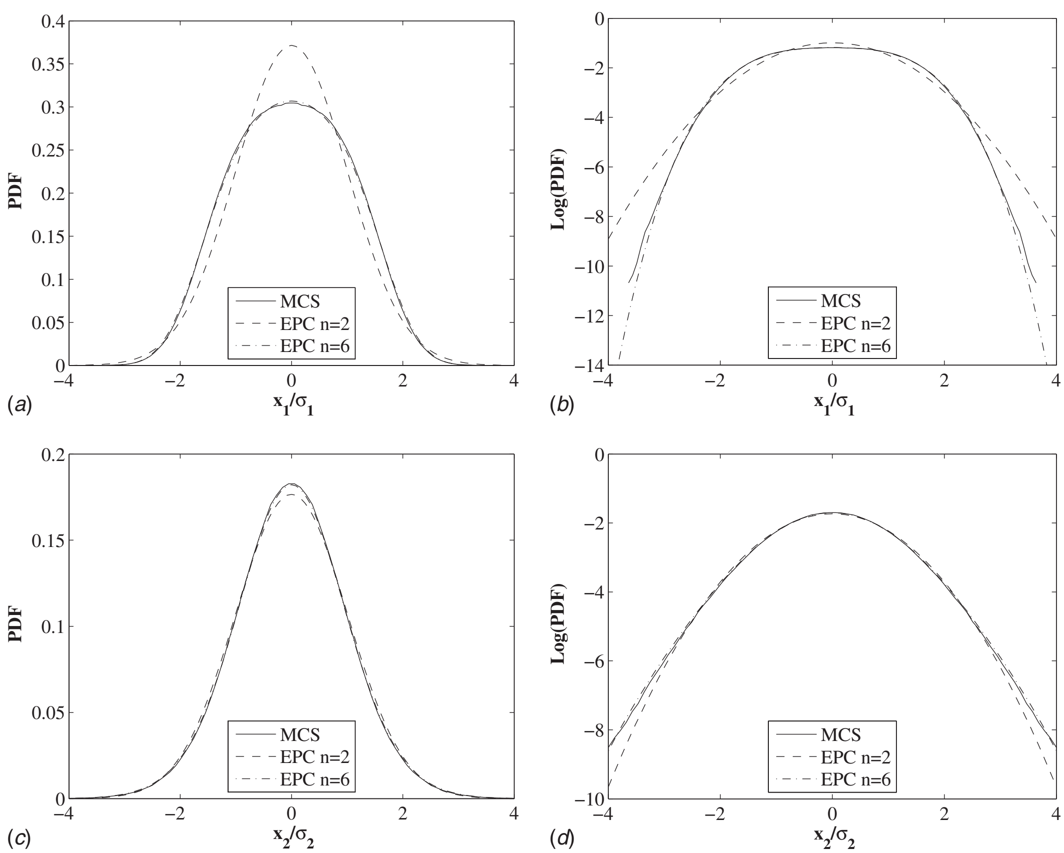

Fig. 2 Comparison of PDFs in Sec. 3.2: (*a*) PDFs of displacement, (*b*) logarithmic PDFs of displacement, (*c*) PDFs of velocity, and (*d*) logarithmic PDFs of velocity

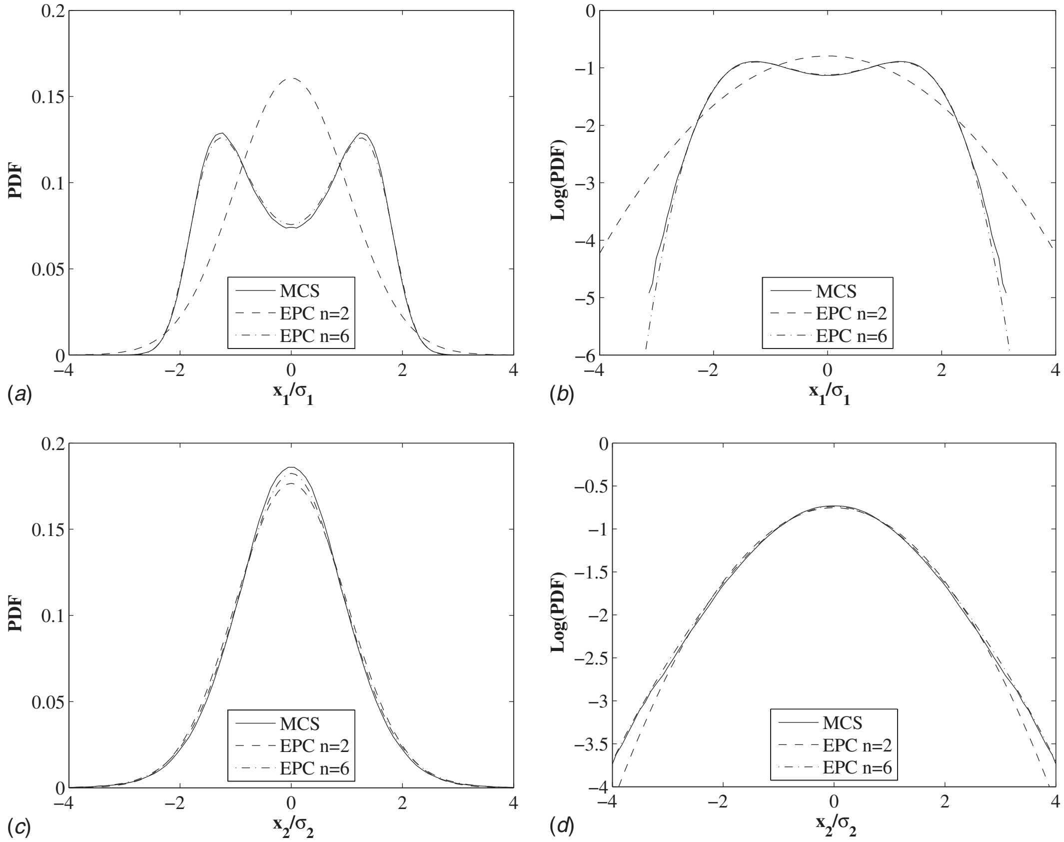

Fig. 3 Comparison of PDFs in Sec. 3.3: (*a*) PDFs of displacement, (*b*) logarithmic PDFs of displacement, (*c*) PDFs of velocity, and (*d*) logarithmic PDFs of velocity

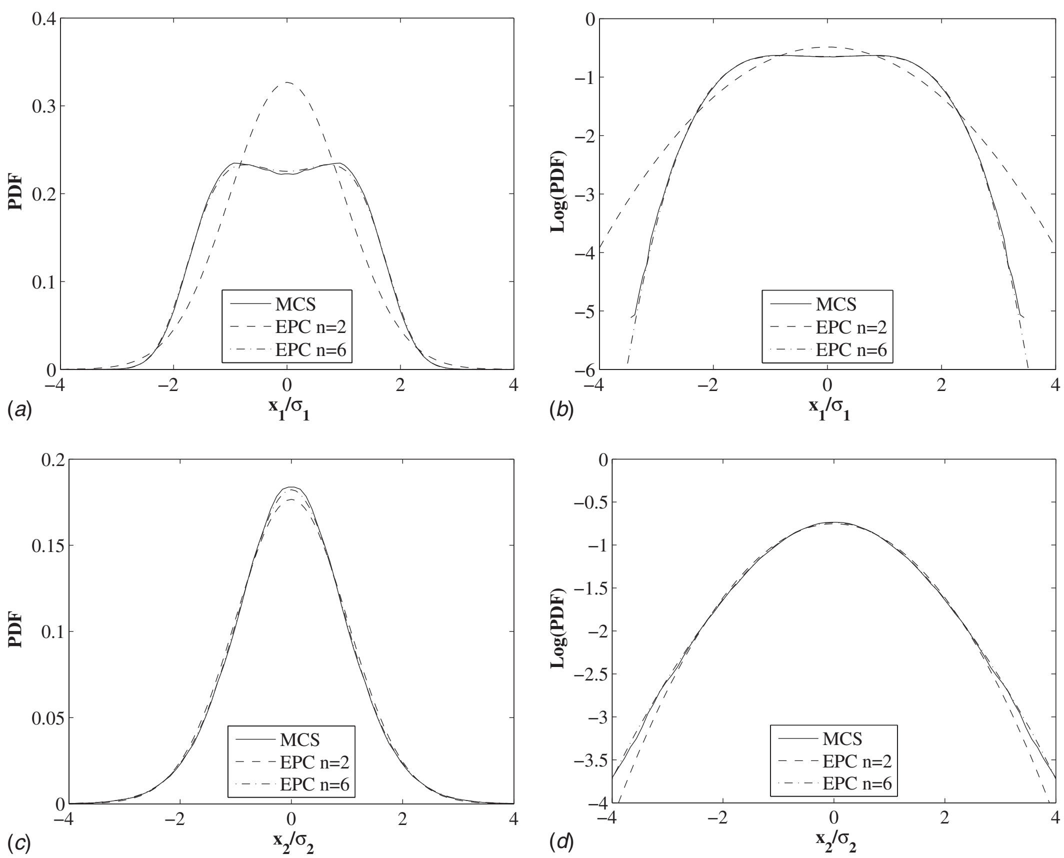

Fig. 4 Comparison of PDFs in Sec. 3.4: (*a*) PDFs of displacement, (*b*) logarithmic PDFs of displacement, (*c*) PDFs of velocity, and (*d*) logarithmic PDFs of velocity

pattern and is in good agreement with the simulation result. This agreement is still observed in the tail regions as shown in Fig. 3(*b*). In this case, the PDF of velocity obtained with EPC ($n=6$) is also improved as shown in Figs. 3(*c*) and 3(*d*).

3.4 Bimodal PDF and Strongly Nonlinear Restoring Force. An oscillator with the bimodal PDF of displacement is further investigated in the case of strong nonlinearity by setting $\alpha=-1$ and $\varepsilon_1=1$. In this case, the PDF of displacement obtained with EPC ($n=6$) is still much improved compared with the result obtained with EPC ($n=2$) or the EQL method as shown in Figs. 4(*a*) and 4(*b*). The PDF of velocity obtained with EPC ($n=6$) is also improved as compared with the result from EPC ($n=2$) or the EQL method.

4 Conclusions

The EPC method originally developed for the PDF solution of nonlinear oscillators under Gaussian excitation is extended for the PDF solution of nonlinear oscillators under both external Poisson white noise excitation and multiplicative Poisson white noise excitation on velocity. From the above discussion and numerical analysis, it is seen that the EPC method of n being 6 can provide much improved PDFs of both displacement and velocity of the nonlinear oscillators no matter if the PDF has the unimodal pattern or bimodal pattern. The PDFs of the responses obtained with the EPC method of n being 6 are in good agreement with the results obtained from Monte Carlo simulation, especially in the tail regions of the PDFs, which is important in reliability analysis. The numerical analysis also shows that the PDFs obtained with the EPC method of n being 2 are close to those obtained with EQL method. Numerical results further show that the PDFs obtained with the EPC method of n being 2 or the EQL method deviate significantly from the PDFs obtained from Monte Carlo simulation not only in the case of strongly nonlinear oscillator but also in the case of weakly nonlinear oscillator, especially in the tail regions of the PDFs of oscillator responses.

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Appendix

The derivatives of the approximate PDF $\tilde{p}(x_1,x_2;\mathbf{a})$ (Eq. (8)) with respect to x_1 and x_2 are given as follows:

$$\frac{\partial\tilde{p}}{\partial x_1}=\frac{\partial Q_n}{\partial x_1}\tilde{p}$$

$$\frac{\partial\tilde{p}}{\partial x_2}=\frac{\partial Q_n}{\partial x_2}\tilde{p}$$

$$\frac{\partial^2\tilde{p}}{\partial x_2^2}=\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^2+\frac{\partial^2 Q_n}{\partial x_2^2}\right]\tilde{p}$$

$$\frac{\partial^3\tilde{p}}{\partial x_2^3}=\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^3+3\frac{\partial Q_n}{\partial x_2}\frac{\partial^2 Q_n}{\partial x_2^2}+\frac{\partial^3 Q_n}{\partial x_2^3}\right]\tilde{p}$$

$$\frac{\partial^4\tilde{p}}{\partial x_2^4}=\left[\left(\frac{\partial Q_n}{\partial x_2}\right)^4+6\left(\frac{\partial Q_n}{\partial x_2}\right)^2\frac{\partial^2 Q_n}{\partial x_2^2}+3\left(\frac{\partial^2 Q_n}{\partial x_2^2}\right)^2+4\frac{\partial Q_n}{\partial x_2}\frac{\partial^3 Q_n}{\partial x_3^3}+\frac{\partial^4 Q_n}{\partial x_2^4}\right]\tilde{p} \tag{A1}$$

Given that Y and Y' are Gaussian distributed with zero mean and the above relations, the residual error expressed by Eq. (11) can be further expanded as shown below.

$$\begin{aligned}\Delta(x_1,x_2;\mathbf{a}) &= -x_2\frac{\partial\tilde{p}}{\partial x_1}+\frac{\partial}{\partial x_2}\left\{\left(h_0-\frac{1}{2}\lambda_2 E[Y'^2]\varepsilon_2^2 x_2\right)\tilde{p}\right\}\\ &\quad+\frac{1}{2!}\lambda_1 E[Y^2]\frac{\partial^2\tilde{p}}{\partial x_2^2}+\frac{1}{4!}\lambda_1 E[Y^4]\frac{\partial^4\tilde{p}}{\partial x_2^4}\\ &\quad+\frac{1}{2!}\lambda_2 E[Y'^2]\varepsilon_2^2\left\{2\tilde{p}+4x_2\frac{\partial\tilde{p}}{\partial x_2}+x_2^2\frac{\partial^2\tilde{p}}{\partial x_2^2}\right\}\\ &\quad+\frac{1}{4!}\lambda_2 E[Y'^4]\varepsilon_2^4\left\{24\tilde{p}+96x_2\frac{\partial\tilde{p}}{\partial x_2}+72x_2^2\frac{\partial^2\tilde{p}}{\partial x_2^2}\right.\\ &\quad\left.+16x_2^3\frac{\partial^3\tilde{p}}{\partial x_2^3}+x_2^4\frac{\partial^4\tilde{p}}{\partial x_2^4}\right\}\end{aligned} \tag{A2}$$

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